STUDENT NAME:

Linear Algebra Graduate Comprehensive Exam, August 2016
Worcester Polytechnic Institute

Work out at least 6 of the following problems
Write down detailed proofs of every statement you make.
No Books. No Notes. No calculators.

1. Show that zero is an eigenvalue of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 5 \end{bmatrix}$$

What is its multiplicity? Does the matrix have nonzero eigenvalues? If so, compute their product. If not, explain why not.

2. Let $L : \mathbb{R}^n \to \mathbb{R}^n$be a linear transformation and for every $k \in \mathbb{N}$, denote by L^k the composition of L with itself k times.

   (i) Show that for every $k \in \mathbb{N}$, one has

$$Range(L^{k+1}) \subseteq Range(L^k)$$

   (ii) Show that there exists a positive integer m such that for all $k \geq m$ one has

$$Range(L^k) = Range(L^{k+1})$$

   (iii) For m as in (ii) set $W = Range(L^m)$. What can you deduce about the restriction of the map L to the subspace W?

3. For every two continuous functions $f, g \in C(\mathbb{R})$ define their product to be

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx.$$

Show that this defines an inner product on the space of all continuous functions $C(\mathbb{R})$.

4. Consider the space V_3 of polynomials with real coefficients of degree at most 3. Find a basis for V_3 such that

   (i) every element of the basis satisfies $p(1) = 1$

   (ii) any two elements in the basis have different degrees

   (iii) any two elements in the basis are orthogonal with respect to the scalar product introduced in the previous problem.

5. Consider $T : \mathbb{P}_3 \to \mathbb{P}_1$ defined by second differentiation, i.e., by $T(p) = p'' \in \mathbb{P}_1$ for $p \in \mathbb{P}_3$. Find the matrix representation of T with respect to the bases
   $\{1 + x, 1 - x, x + x^2, x^2 - x^3\}$ for $\mathbb{P}_3$ and $\{1, x\}$ for $\mathbb{P}_1$.

6. Consider the following matrix

$$A = \frac{1}{2} \begin{bmatrix} 0 & -3 & 0 & 1 \\ -3 & 0 & -1 & 0 \\ 0 & 1 & 0 & 5 \\ -1 & 0 & 5 & 0 \end{bmatrix}$$

   (i) Find the transformation matrix M, and its inverse, such that $J = MAM^{-1}$ is the Jordon canonical form of A.

   (ii) What is the characteristic polynomial of the matrix A?

   (iii) Find all possible Jordan canonical forms of a matrix with the same characteristic polynomial as A.

7. Consider the system $Ax = b$, where A is $m \times n$ with $m < n$ and $\text{rank}(A) = m$.

   (i) Describe the set of all solutions of the system.

   (ii) Show that (AA^T) is nonsingular.

   (iii) Show that $A^T(AA^T)^{-1}b$ is the minimum-norm solution of the system.